

MEASURE AND COCYCLE RIGIDITY FOR CERTAIN NON-UNIFORMLY HYPERBOLIC ACTIONS OF HIGHER RANK ABELIAN GROUPS

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ABSTRACT. We prove absolute continuity of “high entropy” hyperbolic invariant measures for smooth actions of higher rank abelian groups assuming that there are no proportional Lyapunov exponents. For actions on tori and infranilmanifolds existence of an absolutely continuous invariant measure of this kind is obtained for actions whose elements are homotopic to those of an action by hyperbolic automorphisms with no multiple or proportional Lyapunov exponents. In the latter case a form of rigidity is proved for certain natural classes of cocycles over the action.

1. INTRODUCTION

In this paper we continue the program of studying hyperbolic measures for actions of higher rank abelian groups first alluded to in [6, Part II] and started in earnest [7, 11, 9]. We refer to those papers for basic definitions and standard facts concerning those actions.

Specifically we extend some of the principal results of [7, 11, 9] from maximal rank actions (\mathbb{Z}^k actions on $k+1$ -dimensional manifolds and \mathbb{R}^k actions on $2k+1$ -dimensional manifolds, $k \geq 2$) to a class of actions where dimension and rank are not related, except of the standard assumption of rank being at least 2. Thus we partially realize the “Low rank and high dimension” program of [9, Section 8.3.]. While we use the general methods and some specific results from the previous papers as well as heavy machinery of smooth ergodic theory, we introduce three important new ingredients that make these advances possible. These new elements are:

- The *holonomy invariance* that appears in the proof of Theorem 2.8. It is an extension to the general non-linear and non-uniformly hyperbolic actions of certain arguments which appeared in [6] for the study of invariant measures for linear actions on the torus.
- New *entropy inequality*, Lemma 6.1, that is crucial in the proof of Theorem 2.4, by allowing to show that all Lyapunov hyperplanes for the linear action persist for the non-linear one.

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- The *index argument* in the uniqueness proof (Section 7.1, Lemma 7.4) that replaces the argument in [11] that depends on existence of elements with codimension one stable foliations.

Our second goal is to prove cocycle rigidity for actions on the torus satisfying our measure rigidity results. For the case of maximal rank actions those results have been announced in [8].

In this paper we restrict ourselves to the case where all Lyapunov exponents are simple and there are no proportional Lyapunov exponents. This allows to avoid extra technical complications that appear in the presence of multiple or proportional exponents. In this situation coarse Lyapunov foliations are one-dimensional and invariant geometric structures on their leaves are affine. Our approach extends to certain cases where multiple or positively proportional Lyapunov exponents are allowed (totally non-symplectic condition, TNS for short). In this case one needs to use a version of the theory of non-stationary normal forms (see [6, Section 6]) to produce invariant geometric structures on the leaves of coarse Lyapunov foliations. Those structures may be more complicated than affine if there are resonances between Lyapunov exponents. We discuss this more general situation in the last section. Detailed treatment will appear in a separate paper.

2. FORMULATION OF RESULTS

Let α be an \mathbb{R}^k , $k \geq 2$, $C^{1+\theta}$, ($\theta > 0$) action on an n -dimensional manifold M and μ be an invariant ergodic measure for α . Let $\chi_1, \dots, \chi_n; \mathbb{R}^k \rightarrow \mathbb{R}$ be the Lyapunov exponents (linear functionals) associated to μ . Recall that an ergodic invariant measure μ for a smooth locally free \mathbb{R}^k action α is called *hyperbolic* if all nontrivial Lyapunov exponents χ_i , $i = 1, \dots, l$, are nonzero linear functionals on \mathbb{R}^k . Kernels of non-zero Lyapunov exponents are called *Lyapunov hyperplanes*. Vectors in \mathbb{R}^k which do not lie on any of the Lyapunov hyperplanes are called *regular*. Connected components of the sets of regular vectors are called *Weyl chambers*.

Recall that in the absence of positively proportional Lyapunov exponents every Lyapunov distribution E_i integrates to an invariant family of smooth manifolds W^i defined μ a.e. which is customarily called the Lyapunov foliation. Leaves of those foliations are intersections of stables manifolds of properly chosen elements of the action. See Section 3 for details.

2.1. Strongly simple actions.

Definition 1. We say that (α, μ) (or simply α if μ is understood) is *strongly simple* if coarse Lyapunov distributions E^i are one dimensional. Equivalently, all Lyapunov exponents are simple and there are no proportional Lyapunov exponents.

We say that (α, μ) satisfies the *full entropy condition* if the entropy function is not differentiable at Lyapunov hyperplanes.

Theorem 2.1. *Let μ be an ergodic invariant measure for a strongly simple action α . Then for any element \mathbf{t} of the action such that the entropy $h_\mu(\mathbf{t}) > 0$, there exists a Lyapunov exponent χ such that $\chi(\mathbf{t}) < 0$ and conditional measures on the Lyapunov foliation \mathcal{W} corresponding to χ are equivalent to Lebesgue measure.*

Since all Lyapunov exponents change sign for an inverse transformation while the entropy remains the same, Theorem 2.1 immediately implies the following result.

Corollary 2.2. *Let μ be an ergodic invariant measure for a strongly simple action α . If $h_\mu(\mathbf{t}) > 0$ for some $\mathbf{t} \in \mathbb{R}^k$ then there are at least two Lyapunov foliations such that the corresponding conditional measures are equivalent to Lebesgue measure.*

It is probable that one can strengthen Theorem 2.1 in the following way.

Conjecture. *Conditional measures on unstable manifolds of the action elements are equivalent to Lebesgue measures on certain smooth submanifolds that are obtained by integrating those Lyapunov foliations for which conditional measures are Lebesgue.*

The full entropy condition leads to a stronger assertion.

Theorem 2.3. *Let μ be an ergodic invariant measure for an action α . Assume that*

- (1) (α, μ) is strongly simple;
- (2) (α, μ) satisfies the full entropy condition.

Then μ is absolutely continuous (with respect to the smooth measure class on M).

2.2. Actions on tori and nilmanifolds. Let N be a simply connected nilpotent Lie group and A a group of affine transformations of N acting freely that contains a finite index subgroup Γ of translations that is a lattice in N . Then the orbit space N/A is a compact manifold that is called an *infrnilmanifold*. An automorphism of N that maps orbits of A onto orbits of A generates a diffeomorphism of N/A that is called an *infrnilmanifold automorphism*.

An action α_0 of \mathbb{Z}^k by automorphisms of an infrnilmanifold M is an Anosov action if induced linear action on the Lie algebra \mathfrak{N} of N has non-zero Lyapunov exponents.

Now let α be an action of \mathbb{Z}^k by diffeomorphisms of M such that its elements are homotopic to elements of an Anosov action by automorphisms. We will say that α has *homotopy data* α_0 . Recall that there is a unique continuous map $h : M \rightarrow M$ homotopic to identity such that

$$h \circ \alpha = \alpha_0 \circ h.$$

This map is customarily called the semi-conjugacy between α and α_0 .

Let us call an α -invariant Borel probability measure μ *large* if the push-forward $h_*\mu$ is Haar measure. The following theorem is the extension of (Theorems 1.3, and 1.7) from [7] from actions on a torus with Cartan homotopy data to our case of actions on infranilmanifolds with strongly simple homotopy data.

Theorem 2.4. *Let α_0 be a strongly simple Anosov action of \mathbb{Z}^k by automorphisms of an infranilmanifold and let α be a smooth action with homotopy data α_0 . Let μ be an ergodic large invariant measure μ for α . Then:*

- (1) μ is absolutely continuous.
- (2) Lyapunov characteristic exponents of the action α with respect to μ are equal to the Lyapunov characteristic exponents of the action α_0 .

Conjecture. *Under the assumptions of Theorem 2.4 large invariant measure is unique.*

We can prove this under a stronger assumption on α_0 that forces the group N be abelian and hence M to have a torus as a finite cover.

A relation between different Lyapunov exponents χ_1, \dots, χ_l of a \mathbb{Z}^k action of the form $\chi_1 = \sum_{i=2}^l m_i \chi_i$ where m_2, \dots, m_l are positive integers is called a *resonance*. A resonance with $l = 2$, is called a *double resonance*.

In this setting we may assume that $N = \mathbb{R}^m$ and $\Gamma = \mathbb{Z}^m$. The factor of the torus $\mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m$ by finite group of fixed point free affine maps is called an *infratorus*. The following theorem is a generalization of the main result (Corollary 2.2) of [11].

Theorem 2.5. *Let α_0 be a linear strongly simple \mathbb{Z}^k action without resonances on an infratorus. Any action α with homotopy data α_0 has unique large invariant measure μ . Furthermore, the semiconjugacy h is bijective μ a.e. and effects a measurable isomorphism between (α, μ) and α_0 with Haar measure.*

Remark 1. Difference between the Cartan condition of [7] and our strongly simple condition is quite considerable. Cartan actions are maximal rank actions on the torus by hyperbolic automorphisms; thus the rank is equal to the dimension minus one. On the other hand, strongly simple actions may have any rank starting from two: for example, the restriction as a Cartan action to any \mathbb{Z}^2 subgroup is strongly simple. Also strongly simple actions may be reducible; e.g. the product of two Cartan actions is usually strongly simple.

Remark 2. Actions by automorphisms of non-abelian simply connected Lie groups always have resonances that appear from the non-trivial bracket relations in the Lie algebra. While the easiest standard examples have double resonances there are strongly simple actions on some compact nilmanifolds.

Generally speaking, cocycle rigidity means that one-cocycles of certain regularity over a group action are cohomologous to constant cocycles via

transfer functions of certain (often lower) regularity. Cocycle rigidity is prevalent in hyperbolic and partially hyperbolic actions of higher rank abelian groups, see e.g. [10, 12, 13, 3]. The method of [10] for uniformly hyperbolic actions satisfying TNS condition (no negatively proportional Lyapunov exponents) can be used in the non-uniformly hyperbolic case and is in particular applicable in the settings considered in the present paper.

One can apply this method to smooth or Hölder continuous cocycles but there are reasons to consider two broader classes of cocycles which are in general defined only almost everywhere with respect to a hyperbolic absolutely continuous invariant measure:

- (i) *Lyapunov Hölder* cocycles: Hölder continuous with respect to a properly defined *Lyapunov metric* which is equivalent to a smooth metric on Pesin sets and changes slowly along the orbits, and
- (ii) *Lyapunov smooth* cocycles: smooth along invariant foliations at the points of Pesin sets with a similar slow change condition.

See Section 8 for a more detailed formal description of those classes of cocycles. Notice that the most important intrinsically defined cocycles, the logarithms of the Jacobians along invariant foliations (Lyapunov, stable, and likewise), are Lyapunov smooth.

Theorem 2.6. *For any action α of \mathbb{Z}^k on an infratorus as in Theorem 2.5 any Lyapunov Hölder (corr. Lyapunov smooth) cocycle is cohomologous to a constant cocycle via a Lyapunov Hölder (corr. Lyapunov smooth) transfer function.*

Remark 3. Assertion of this theorem is likely to hold in the more general setting of Theorem 2.4. The difficulties in the proof are of technical nature and have to do with a proper application of a version of the Hopf argument in the resonance case when invariant geometric structures on stable foliations are not affine.

2.3. Technical results and overall structure of proofs.

2.3.1. The starting point in the proofs of all three results formulated above, Theorems 2.1, 2.3 and 2.4 is the Main technical Theorem we proved in [9, Theorem 4.1] joint with B. Kalinin. So, let us recall it.

Theorem 2.7. *Let μ be a hyperbolic ergodic invariant measure for a locally free $C^{1+\theta}$, $\theta > 0$, action α of \mathbb{R}^k , $k \geq 2$, on a compact smooth manifold M . Suppose that a Lyapunov exponent χ is simple and there are no other exponents proportional to χ . Let E be the one-dimensional Lyapunov distribution corresponding to the exponent χ and \mathcal{W} the corresponding Lyapunov foliation.*

Then conditional measures of μ on \mathcal{W} are either atomic a.e. or equivalent to Lebesgue measure a.e.

2.3.2. Let α and χ be as in the hypothesis of technical Theorem 2.7. Let us fix a generic singular element $\mathbf{t} \in \mathbb{R}^k$, i.e. an element such that $\chi(\mathbf{t}) = 0$ but $\chi_j(\mathbf{t}) \neq 0$ for any other Lyapunov exponent, and an element \mathbf{s} close to \mathbf{t} such that $\chi(\mathbf{s}) < 0$ but it is still the biggest negative Lyapunov exponent for \mathbf{s} . Then we have that $\mathcal{W}_{\alpha(\mathbf{t})}^s \subset \mathcal{W}_{\alpha(\mathbf{s})}^s$ and in fact $E_{\alpha(\mathbf{s})}^s = E_{\alpha(\mathbf{t})}^s \oplus E_\chi$.

The second principal technical result that appears in the proofs of Theorems 2.3 directly and 2.4 (through Theorem 2.9) is new. It shows that if we have atomic conditional measures along some \mathcal{W}^i direction then conditional measures along stable manifolds containing \mathcal{W}^i direction fit in a lower dimensional submanifold.

Theorem 2.8. *If χ, \mathbf{t} and \mathbf{s} are as above and the conditional measure along \mathcal{W} is atomic for a.e. point then the conditional measure along $\mathcal{W}_{\alpha(\mathbf{s})}^s(x)$ has its support inside $\mathcal{W}_{\alpha(\mathbf{t})}^s(x)$ for a.e. x .*

While the proof of Theorem 2.8 does not use Theorem 2.7 directly it relies on the the principal technical construction that appears in the proof of the latter result: the synchronizing time change, see Section 3.3.

Theorems 2.7 and 2.8 are also used in the proof of the following result that is used in the proof of Theorem 2.3.

Theorem 2.9. *Let χ be as in Theorem 2.7 then conditional measure along \mathcal{W} is Lebesgue if entropy function is not differentiable at the Lyapunov hyperplane $\ker \chi$.*

2.3.3. In addition to innovations listed in the Introduction other additional major ingredients that appear in the proofs are:

- (i) Ledrappier-Young entropy formula reviewed in Section 3.4 and used in proofs of Theorems 2.9 and 2.3 and
- (ii) an extension of H.Hu's result on linearity of the entropy functions inside Weyl chambers, see 3.9. It is used in the proof of Theorem 2.9 to treat a case when a Lyapunov exponent is proportional to the difference of two other, that may appear in strongly simple actions.

Remark 4. It is probable that the converse to the statement of Theorem 2.9 is also true. This would follow from an extension of the Ledrappier-Young formula, see statement (A) in Section 3.4.

3. PRELIMINARIES

3.1. Lyapunov exponents and Pesin sets. For a smooth \mathbb{R}^k action α on a manifold M and an element $\mathbf{t} \in \mathbb{R}^k$ we denote the corresponding diffeomorphism of M by $\alpha(\mathbf{t})$. Sometimes we will omit α and write, for example, $\mathbf{t}x$ in place of $\alpha(\mathbf{t})x$ and $D\mathbf{t}$ in place of $D\alpha(\mathbf{t})$ for the derivative of $\alpha(\mathbf{t})x$.

Proposition 3.1. *Let α be a locally free $C^{1+\theta}$, $\theta > 0$, action of \mathbb{R}^k on a manifold M preserving an ergodic invariant measure μ . There are linear*

functionals χ_i , $i = 1, \dots, l$, on \mathbb{R}^k and an α -invariant measurable splitting of the tangent bundle TM , called the Lyapunov decomposition, (or sometimes the Oseledets decomposition), $TM = T\mathcal{O} \oplus \bigoplus_{i=1}^l E_i$ over a set of full measure \mathfrak{Rc} , where $T\mathcal{O}$ is the distribution tangent to the \mathbb{R}^k orbits, such that for any $\mathbf{t} \in \mathbb{R}^k$ and any nonzero vector $v \in E_i$ the Lyapunov exponent of v is equal to $\chi_i(\mathbf{t})$, i.e.

$$\lim_{n \rightarrow \pm\infty} n^{-1} \log \|D(n\mathbf{t})v\| = \chi_i(\mathbf{t}),$$

where $\|\cdot\|$ is any continuous norm on TM . Any point $x \in \mathfrak{Rc}$ is called a regular point.

Furthermore, for any $\varepsilon > 0$ there exist positive measurable functions $C_\varepsilon(x)$ and $K_\varepsilon(x)$ such that for all $x \in \mathfrak{Rc}$, $v \in E_i(x)$, $\mathbf{t} \in \mathbb{R}^k$, and $i = 1, \dots, l$,

- (1) $C_\varepsilon^{-1}(x)e^{\chi_i(\mathbf{t}) - \frac{1}{2}\varepsilon\|\mathbf{t}\|}\|v\| \leq \|D\mathbf{t}v\| \leq C_\varepsilon(x)e^{\chi_i(\mathbf{t}) + \frac{1}{2}\varepsilon\|\mathbf{t}\|}\|v\|$;
- (2) Angles $\angle(E_i(x), T\mathcal{O}) \geq K_\varepsilon(x)$ and $\angle(E_i(x), E_j(x)) \geq K_\varepsilon(x)$, $i \neq j$;
- (3) $C_\varepsilon(\mathbf{t}x) \leq C_\varepsilon(x)e^{\varepsilon\|\mathbf{t}\|}$ and $K_\varepsilon(\mathbf{t}x) \geq K_\varepsilon(x)e^{-\varepsilon\|\mathbf{t}\|}$.

The stable and unstable distributions $E_{\alpha(\mathbf{t})}^s$ and $E_{\alpha(\mathbf{t})}^u$ of an element $\alpha(\mathbf{t})$ are defined as the sums of the Lyapunov distributions corresponding to the negative and the positive Lyapunov exponents for $\alpha(\mathbf{t})$ respectively. Notice that stable and unstable distributions are the same within a Weyl chamber, and conversely, the set of vectors with given stable and unstable distributions, if non-empty, is a Weyl chamber. A minimal non-zero intersection of stable distributions for various elements of the action is called a *coarse Lyapunov distribution*. Equivalently, any coarse Lyapunov distribution is the sum of Lyapunov distributions corresponding all Lyapunov exponents positively proportional to each other. Notice that in the absence of positively proportional Lyapunov exponents, in particular, for the strongly simple case considered in this paper, coarse Lyapunov distributions coincide with Lyapunov distributions.

3.2. Invariant manifolds and affine structures. We will use standard material on invariant manifolds corresponding to the negative and positive Lyapunov exponents (stable and unstable manifolds) for $C^{1+\theta}$ measure preserving diffeomorphisms of compact manifolds, see for example [1, Chapter 4]. In particular, stable distributions and hence their transversal intersections are always Hölder continuous (see, for example, [2]). Here is a summary of some of those results adapted to the case of an \mathbb{R}^k action.

Proposition 3.2. *Let α be a $C^{1+\theta}$, $\theta > 0$ action of \mathbb{R}^k as in Proposition 3.1. Suppose that a Lyapunov distribution E is the intersection of the stable distributions of some elements of the action. Then E is Hölder continuous on any Pesin set*

$$(3.1) \quad \mathfrak{Rc}_\varepsilon^l = \{x \in \mathfrak{Rc} : C_\varepsilon(x) \leq l, K_\varepsilon(x) \geq l^{-1}\}$$

with Hölder constant which depends on l and Hölder exponent $\delta > 0$ which depends on the action α only.

Furthermore on those sets the size of local stable manifolds for any element of α is bounded away from below.

We will denote by $\mathcal{W}_{\alpha(\mathbf{t})}^s(x)$ the (global) stable manifold for $\alpha(\mathbf{t})$ at a regular point x . This manifold is an immersed Euclidean space tangent to the stable distribution $E_{\alpha(\mathbf{t})}^s$. The unstable manifold $\mathcal{W}_{\alpha(\mathbf{t})}^u(x)$ is defined as the stable one for $\alpha(-\mathbf{t})$ and thus have similar properties. Local stable/unstable manifolds will be denoted by $W_{\alpha(\mathbf{t})}^s(x)$ and $W_{\alpha(\mathbf{t})}^u(x)$ correspondingly.

Intersections of stable manifolds for different elements of the action are integral manifolds for coarse Lyapunov distributions and they form a *coarse Lyapunov foliations*. While in general Lyapunov foliations may not be uniquely integrable this is obviously the case in the absence of positively proportional Lyapunov exponents. Hence everywhere in this paper we will talk about *Lyapunov foliations*. These foliations are defined for any Lyapunov distribution E as in Proposition 3.2. We will denote the Lyapunov foliation corresponding to the exponent χ by \mathcal{W} and its local leaf at a regular point x by $W(x)$.

As a comment on terminology let us emphasize that it is customary to use words “distributions” and “foliations” in this setting although these objects are correspondingly measurable families of tangent spaces defined a.e. and measurable families of smooth manifolds which fill a set of full measure.

Let us also recall the existence of affine structures. Let α be an action as in Theorem 2.7. The following proposition provides α -invariant affine parameters on the leaves of the Lyapunov foliation \mathcal{W} .

Proposition 3.3. [7, Proposition 3.1., Remark 5] *There exist a unique family of $C^{1+\theta}$ smooth α -invariant affine parameters on the leaves $\mathcal{W}(x)$. Moreover, they depend uniformly continuously in $\mathbb{C}^{1+\theta}$ topology on x in any Pesin set.*

3.3. Lyapunov metrics and synchronizing time change. We fix a smooth Riemannian metric $\langle \cdot, \cdot \rangle$ on M . Given $\varepsilon > 0$ and a regular point $x \in M$ we define the standard ε -Lyapunov scalar product (or metric) $\langle \cdot, \cdot \rangle_{x,\varepsilon}$ as follows. For any $u, v \in E(x)$ we define

$$(3.2) \quad \langle u, v \rangle_{x,\varepsilon} = \int_{\mathbb{R}^k} \langle (Ds)u, (Ds)v \rangle \exp(-2\chi(\mathbf{s}) - 2\varepsilon\|\mathbf{s}\|) d\mathbf{s}.$$

We shall need to use the time change we introduced with B. Kalinin in [9] in the context of Theorem 2.7. Let $L = \ker \chi$, fix a vector $\mathbf{w} \in \mathbb{R}^k$ normal to L with $\chi(\mathbf{w}) = 1$ and take $\varepsilon > 0$ small such that $\varepsilon\|\mathbf{w}\|$ is also small, in particular less than $1/2$.

Proposition 3.4. [9, Proposition 6.2, Proposition 6.3] *For μ -a.e. x and any $\mathbf{t} \in \mathbb{R}^k$ there exists $g(x, \mathbf{t}) \in \mathbb{R}^k$ such that the function $\mathbf{g}(x, \mathbf{t}) = \mathbf{t} + g(x, \mathbf{t})\mathbf{w}$ satisfies the equality*

$$\|D_x^E \alpha(\mathbf{g}(x, \mathbf{t}))\|_\varepsilon = e^{\chi(\mathbf{t})}.$$

The function $g(x, \mathbf{t})$ is measurable and is Hölder continuous on Pesin sets, is C^1 in \mathbf{t} and $|g(x, \mathbf{t})| \leq 2\varepsilon \|\mathbf{t}\|$. Moreover, the formula $\beta(\mathbf{t}, x) = \alpha(\mathbf{g}(x, \mathbf{t}))x$ defines an \mathbb{R}^k action β on M which is a measurable time change of α . The action β is measurable and continuous on Pesin sets for α and preserves a measure ν which is equivalent to μ .

Now we describe invariant "foliations" for β whose leaves are not smooth but still these objects have properties close to true invariant foliations for smooth actions. Let us denote with \mathcal{N} the orbit foliation of the one-parameter subgroup $\{r\mathbf{w}\}$.

Proposition 3.5. *For any element $\mathbf{s} \in \mathbb{R}^k$ there exist a stable "foliation" $\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^{\mathbf{s}}$ which is contracted, by $\beta(\mathbf{s})$ and invariant under the new action β . It consists of "leaves" $\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^{\mathbf{s}}(x)$ defined for every x . The "leaf" $\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^{\mathbf{s}}(x)$ is a measurable subset of the leaf $(\mathcal{N} \oplus \mathcal{W}_{\alpha(\mathbf{s})}^{\mathbf{s}})(x)$ of the form*

$$\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^{\mathbf{s}}(x) = \{\alpha(\phi_x(y)\mathbf{w})y : y \in \mathcal{W}_{\alpha(\mathbf{s})}^{\mathbf{s}}(x)\},$$

where $\phi_x : \mathcal{W}_{\alpha(\mathbf{s})}^{\mathbf{s}}(x) \rightarrow \mathbb{R}$ is an almost everywhere defined measurable function. For x in a Pesin set, the ϕ_x is Hölder continuous on the intersection of this Pesin set with any ball of fixed radius in $\mathcal{W}_{\alpha(\mathbf{s})}^{\mathbf{s}}(x)$ with Hölder exponent γ and constant which depends on the Pesin set and radius.

We will use the fact for any $\mathbf{s} \in \mathbb{R}^k$ the partition into global stable manifolds $\tilde{\mathcal{W}}_{\beta(\mathbf{s})}^{\mathbf{s}}(x)$ refines the partition into ergodic components of $\beta(\mathbf{s})$.

3.4. Ledrappier-Young entropy formula. Let $f : M \rightarrow M$ be $C^{1+\theta}$ diffeomorphism and let μ be an ergodic invariant measure. Let $\chi_1 > \chi_2 > \dots > \chi_r$ be its Lyapunov exponents and let $TM = E_1 \oplus \dots \oplus E_r$ be the corresponding Lyapunov decomposition. Let $u = \max\{i : \chi_i > 0\}$ and for $1 \leq i \leq u$ let us define

$$V^i(x) = \{i \in M \mid \limsup_{n \rightarrow \infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) \leq -\chi_i\}.$$

For a.e. x $V^i(x)$ is a smooth manifold tangent to $\bigoplus_{j \leq i} E_j$ and we have the flag $V^1 \subset V^2 \subset \dots \subset V^u$ with $V^u = W^u$, the unstable manifold. We can build partitions ξ^i subordinated to V^i as the ones built in [15] and consider conditional measures μ_x^i . Let $B^i(x, \varepsilon)$ be the ε ball in $V^i(x)$ centered in x with respect to the induced Riemannian metric. Then

$$\delta_i = \delta_i(f) = \lim_{\varepsilon \rightarrow 0} \frac{\log \mu_x^i B^i(x, \varepsilon)}{\log \varepsilon}$$

exists a.e. and does not depend on x . Moreover, calling $\gamma_i = \gamma_i(f) = \delta_i - \delta_{i-1}$ we have the Ledrappier-Young entropy formula (see [15, Theorem C])

$$(3.3) \quad h_\mu(f) = \sum_{1 \leq j \leq u} \gamma_j \chi_j.$$

In fact, a more precise statement is true. Given a measure μ and two measurable partitions α, β , the conditional entropy is defined by:

$$H(\alpha|\beta) = - \int \log \mu_x^\beta(\alpha(x)) d\mu.$$

Let $T : X \rightarrow X$ be a measure preserving transformation. Given a measurable partition α we define the entropy of T w.r.t. α by $h(\alpha, T) = H(T^{-1}\alpha|\alpha^+)$ where $\alpha^+ = \bigvee_{n \geq 0} T^n \alpha$.

We have for every $1 \leq i \leq u$

$$h_\mu(\xi_i, f) = \sum_{1 \leq j \leq i} \gamma_j \chi_j.$$

The following addition to the Ledrappier-Young formula is needed for the proof of the converse to Theorem 2.9.

(2) If in Ledrappier-Young formula (3.3) $\gamma_i = 0$, i.e.

$$h_\mu(\xi_i, f) = h_\mu(\xi_{i-1}, f)$$

then the conditional measure on an almost every leaf of V^i is supported on a single leaf of V^{i-1}

While this statement looks completely natural and is likely to be true, existing arguments and constructions do not yield a proof.

3.5. Linearity of entropies. Given two commuting diffeomorphisms f and g , preserving a measure μ , with coinciding unstable manifolds, in [5], H. Hu built a partition subordinated to this unstable manifold which is increasing for both maps. The same construction may be carried out for partitions subordinated to simultaneous fastest directions, following the same lines, to get the following analogous of Proposition 8.1. in [5].

Let f and g be two diffeomorphisms preserving a measure μ . Assume both maps preserve a measurable bundle $F \subset TM$ such that Lyapunov exponents associated to F , for both f and g , are larger than any other Lyapunov exponents. Hence we get that F is tangent to a "foliation" V which is exactly the fastest foliation associated to the first $\dim(F)$ exponents.

Proposition 3.6. *There is a measurable partition η on M with the following properties.*

- (1) η is subordinated to V .
- (2) η is increasing for f and g , i.e. $f\eta < \eta$ and $g\eta < \eta$.
- (3) $\bigvee_{n \geq 0} f^{-n}\eta$ and $\bigvee_{n \geq 0} g^{-n}\eta$ are the partitions into points.
- (4) The biggest σ -algebra contained in $\bigcap_{n \geq 0} \bigcap_{k \geq 0} f^{-n}g^{-k}\eta$ is the σ -algebra of sets saturated by leaves of V .

As always, (3) and (4) follows from (1) and (2). Also, as in [14], we have that the entropy of f and g w.r.t. this partition does not depend on the partition as long as the partition is subordinated to V , i.e.

Lemma 3.7. *If η and $\hat{\eta}$ are two partitions subordinated to V as in Proposition 3.6, then $h_\mu(\eta, f) = h_\mu(\hat{\eta}, f)$. The same holds for f , g and $f \circ g$.*

Thus, we can call $h_\mu(V, f) = h_\mu(\eta, f)$.

This gives as the analogous of Proposition 9.1. in [5] which gives that entropy is linear.

Proposition 3.8. *Let f , g , μ , V and η be as above. Then*

$$h_\mu(V, f \circ g) = h_\mu(V, f) + h_\mu(V, g).$$

Since the proof is in two lines we repeat it here, let us write fg for $f \circ g$.

Proof.

$$\begin{aligned} h_\mu(\eta, fg) &= H(\eta|fg\eta) = H(\eta \vee g\eta|fg\eta) = H(g\eta|fg\eta) + H(\eta|g\eta \vee fg\eta) \\ &= H(\eta|f\eta) + H(\eta|g\eta) = h_\mu(\eta, f) + h_\mu(\eta, g) \end{aligned}$$

□

Corollary 3.9. *If C is the cone where F is still the fastest bundle, then*

$$\mathbf{s} \rightarrow h_\mu(V, \alpha(\mathbf{s}))$$

is linear for $\mathbf{s} \in C$.

4. PROOF OF THEOREMS 2.8 AND 2.9

4.1. Holonomy invariance of conditional measures: A model case.

Before considering the situation that appears in Theorem 2.8 we shall discuss another case of holonomy invariance of conditional measures along stable directions that is of independent interest and that has a similar but simpler proof.

Let f be a diffeomorphism preserving a measurable foliation \mathcal{F} with smooth leaves. Assume also that $Df|T\mathcal{F}$ is an isometry. Let μ be an f -invariant measure and $\mu_x^\mathcal{F}$ be the conditional measure along $\mathcal{F}(x)$. Recall that those measures are defined up to a scalar multiple; we shall use the normalization $\mu_x^\mathcal{F}(B^\mathcal{F}(x, 1)) = 1$, where $B^\mathcal{F}(x, 1)$ is the ball in the leaf $\mathcal{F}(x)$ centered in x of radius 1. Given a regular point x and $y \in W^s(x)$ let us call $h_{xy} : \mathcal{F}(x) \rightarrow \mathcal{F}(y)$ the holonomy along the stable manifolds; if it is clear from the context we shall omit the lower index xy .

Proposition 4.1. *Let f , \mathcal{F} and μ be as above. Then for μ -a.e. x and μ_x^s -a.e. y in $W^s(x)$ we have that $h_*\mu_x^\mathcal{F} = \mu_y^\mathcal{F}$.*

Proof. If x and y are in the same leaf of \mathcal{F} then $\mu_x^\mathcal{F} = c_{xy}\mu_y^\mathcal{F}$ for some positive constant c_{xy} . Also, by invariance of μ and \mathcal{F} and that f restricted to \mathcal{F} -leaves is an isometry we get that $f_*\mu_x^\mathcal{F} = \mu_{f(x)}^\mathcal{F}$.

Consider now $\mu_x^{\mathcal{F},1} = \mu_x^\mathcal{F}|_{B^\mathcal{F}(x,1)}$ the restriction of $\mu_x^\mathcal{F}$ to $B^\mathcal{F}(x,1)$. By invariance of conditional measures and our choice of normalizations we have $f_*\mu_x^{\mathcal{F},1} = \mu_{f(x)}^{\mathcal{F},1}$.

We will prove that for μ -a.e. x and for μ_x^s -a.e. $y \in W^s(x)$

$$h_*\mu_x^{\mathcal{F},1} = \mu_y^{\mathcal{F},1}$$

Take a sequence $n_i \rightarrow \infty$ such that $f^{n_i}(x) \rightarrow z$, and hence $f^{n_i}(y) \rightarrow z$. Since f restricted to the leafs of \mathcal{F} is an isometry we may take a subsequence such that $f^{n_i}|_{\mathcal{F}(x)}$ converges uniformly on compact sets to an isometry $g_{xz} : \mathcal{F}(x) \rightarrow \mathcal{F}(z)$. Similarly we have that $f^{n_i}|_{\mathcal{F}(y)}$ converges uniformly on compact sets to an isometry $g_{yz} : \mathcal{F}(y) \rightarrow \mathcal{F}(z)$ and

$$h = g_{yz}^{-1} \circ g_{xz}$$

since stable manifolds are contracted in the future. Thus it is sufficient to prove that $(g_{xz})_*\mu_x^{\mathcal{F},1} = \mu_z^{\mathcal{F},1}$ and similarly $(g_{yz})_*\mu_y^{\mathcal{F},1} = \mu_z^{\mathcal{F},1}$. But we have that $f_*^{n_i}\mu_x^{\mathcal{F},1} = \mu_{f^{n_i}(x)}^{\mathcal{F},1}$ and that $f^{n_i} \rightarrow g_{xz}$ and $f^{n_i}(x) \rightarrow z$. Hence we have $f_*^{n_i}\mu_x^{\mathcal{F},1} \rightarrow (g_{xz})_*\mu_x^{\mathcal{F},1}$ and $f_*^{n_i}\mu_y^{\mathcal{F},1} \rightarrow (g_{yz})_*\mu_y^{\mathcal{F},1}$. So we need to prove that

$$\mu_{f^{n_i}(x)}^{\mathcal{F},1} \rightarrow \mu_z^{\mathcal{F},1} \quad \text{and} \quad \mu_{f^{n_i}(y)}^{\mathcal{F},1} \rightarrow \mu_z^{\mathcal{F},1}.$$

To this end we need to use some kind of continuity of the map $x \rightarrow \mu_x^{\mathcal{F},1}$. This map is only measurable so we need to do something to guarantee some kind of continuity. We can apply Luzin's theorem and obtain continuity on a compact set of an arbitrary large measure, so we need to pick the iterates (and hence z) inside this set. The problem is to pick *the same* iterates of x and y in this large set and we will now explain how to achieve that.

We have that $x \rightarrow \mu_x^{\mathcal{F},1}$ is a measurable map and so we have by Luzin's theorem an increasing sequence of compact sets K_n , $\mu(K_n) \rightarrow 1$, such that the map restricted to K_n is continuous. Consider $\tilde{\mathbf{1}}_{K_n}$ the forward Birkhoff average of $\mathbf{1}_{K_n}$ the characteristic function of K_n . Take R_n the set of points where $\tilde{\mathbf{1}}_{K_n} > 1/2$; $\mu(R_n) \rightarrow 1$ since $\mu(K_n) \rightarrow 1$. Since the partition into stable manifolds refines the partition into ergodic components of f , for μ -a.e. point $x \in R_n$, μ_x^s a.e. point in $W^s(x)$ is inside R_n . Take one of these typical points $x \in R_n$ and $y \in W^s(x) \cap R_n$. Let $L(x) = \{n \geq 0 : f^n(x) \in K_n\}$ and similarly $L(y) = \{n \geq 0 : f^n(y) \in K_n\}$. We have from the choice of R_n and since $x, y \in R_n$

$$\frac{\#(L(x) \cap [0, n])}{n} \rightarrow \tilde{\chi}_{K_n}(x) > \frac{1}{2}$$

and the same is true for y . Hence both sets $L(x)$ and $L(y)$ have asymptotic density greater than $1/2$ and hence they should intersect in a set of positive asymptotic density, in particular $L(x) \cap L(y)$ is an infinite set. So we take the sequence n_i inside $L(x) \cap L(y)$ and we get then that $f^{n_i}(x)$ and $f^{n_i}(y)$ are inside K_n and hence their limit z is also in K_n . Now by continuity of $x \rightarrow \mu_x^{\mathcal{F},1}$ restricted to K_n we get that $\mu_{f^{n_i}(x)}^{\mathcal{F},1} \rightarrow \mu_z^{\mathcal{F},1}$ and $\mu_{f^{n_i}(y)}^{\mathcal{F},1} \rightarrow \mu_z^{\mathcal{F},1}$. \square

4.2. Proof of Theorem 2.8. Now let us return to our case. We want to prove the same invariance by holonomy of the conditional measures in a less uniform but more specialized setting.

Let α , μ and χ satisfy the assumptions of technical Theorem 2.7. Let us fix a generic singular element $\mathbf{t} \in \mathbb{R}^k$, i.e. an element such that $\chi(\mathbf{t}) = 0$ but $\chi_j(\mathbf{t}) \neq 0$ for any other Lyapunov exponent, and an element \mathbf{s} close to \mathbf{t} such that $\chi(\mathbf{s}) < 0$ but it is still the biggest negative Lyapunov exponent for \mathbf{s} . Then we have that $\mathcal{W}_{\alpha(\mathbf{t})}^s \subset \mathcal{W}_{\alpha(\mathbf{s})}^s$ and in fact $E_{\alpha(\mathbf{s})}^s = E_{\alpha(\mathbf{t})}^s \oplus E_\chi$. We have the invariant foliation \mathcal{W} associated to χ tangent to E_χ and we consider the conditional measures $\mu^\mathcal{W}$ associated to this foliation that we normalize in a certain convenient way. Given x and $y \in \mathcal{W}_{\alpha(\mathbf{t})}^s(x)$ we define the holonomy map $h_{xy} : \mathcal{W}(x) \rightarrow \mathcal{W}(y)$ by sliding along $\mathcal{W}_{\alpha(\mathbf{t})}^s$ manifolds; we omit the lower index xy if it is understood from the context.

Theorem 2.8 is an immediate corollary of the following holonomy invariance property of conditional measures.

Proposition 4.2. *Let χ, \mathbf{t} and μ be as above. Then for μ -a.e. x and μ_x^s -a.e. $y \in \mathcal{W}_{\alpha(\mathbf{t})}^s(x)$, there is a scalar measurable function $c(x, y)$ such that $h_* \mu_x^\mathcal{W} = c(x, y) \mu_y^\mathcal{W}$ where h is holonomy along $\mathcal{W}_{\alpha(\mathbf{t})}^s$.*

Remark 5. Both Propositions 4.1 and 4.2 assert that the system of conditional measures, defined affinely, is holonomy invariant. The difference is that in the former case there is a normalization that makes normalized conditional measures invariant.

Proof. We will argue as in the proof of Proposition 4.1 but we need to address the problem that dynamics of $\alpha(t)$ along \mathcal{W} is not an isometry although the Lyapunov exponent along \mathcal{W} is equal to zero.

By Proposition 3.3 there is a measurable α -invariant family of affine parameters $H_x : \mathbb{R} \rightarrow \mathcal{W}(x)$. We normalize $\mu_x^\mathcal{W}$ in such a way that $\mu_x^\mathcal{W}(H_x(-1, 1)) = 1$ and define $\mu_x^{\mathcal{W},1} = \mu_x^\mathcal{W}|_{H_x(-1, 1)}$.

We shall use the time change β introduced in Proposition 3.4. Using Luzin's Theorem, let K_n be an increasing sequence of compact sets (which we take also inside Pesin sets for α) $\nu(K_n) \rightarrow 1$ for the β -invariant measure ν (and hence $\mu(K_n) \rightarrow 1$) such that on the set K_n

- (1) the time change is continuous,
- (2) the map $x \rightarrow H_x$ is continuous (with $C^1(\mathbb{R}, M)$ topology for affine structures), and
- (3) the map $x \rightarrow \mu_x^{\mathcal{W},1}$ is continuous with weak $*$ topology in measures on $[-1, 1]$.

Take $f = \beta(t)$ and consider $\tilde{\mathbf{1}}_{K_n}$ the forward Birkhoff average of the characteristic function $\mathbf{1}_{K_n}$. Let as before R_n be the set of points where $\tilde{\mathbf{1}}_{K_n} > 1/2$. Since $\nu(K_n) \rightarrow 1$ then $\nu(R_n) \rightarrow 1$ (and hence $\mu(R_n) \rightarrow 1$). Also, since the partition into $\tilde{\mathcal{W}}_{\beta(\mathbf{t})}^s$ stable "manifolds" refines the partition into ergodic components (see the remark after Proposition 3.5) we have that

for ν -a.e. point x in R_n , ν_x^s a.e. point in $\tilde{\mathcal{W}}_{\beta(\mathbf{t})}^s(x)$ is inside R_n . Take one of these typical points $x \in R_n$ and $y \in \tilde{\mathcal{W}}_{\beta(\mathbf{t})}^s(x) \cap R_n$. As in the proof of Proposition 4.1 we can take a sequence of iterates n_i such that $f^{n_i}(x)$ and $f^{n_i}(y)$ are inside K_n and hence their limit z is also in K_n . Now by continuity of $x \rightarrow \mu_x^{\mathcal{W},1}$ restricted to K_n we get that $\mu_{f^{n_i}(x)}^{\mathcal{W},1} \rightarrow \mu_z^{\mathcal{W},1}$ and $\mu_{f^{n_i}(y)}^{\mathcal{W},1} \rightarrow \mu_z^{\mathcal{W},1}$.

On the other hand if we denote $\mathbf{a}_i = \mathbf{g}(x, n_i \mathbf{t})$ and $\mathbf{b}_i = \mathbf{g}(y, n_i \mathbf{t})$ we have that $\|D_x^E \alpha(\mathbf{a}_i)\|_\varepsilon = 1$ and $\|D_y^E \alpha(\mathbf{b}_i)\|_\varepsilon = 1$. Hence, using the affine parameters from Proposition 3.3 we obtain

$$(4.1) \quad \alpha(\mathbf{a}_i) \circ H_x = H_{\alpha(\mathbf{a}_i)(x)} \quad \text{and} \quad \alpha(\mathbf{b}_i) \circ H_y = H_{\alpha(\mathbf{b}_i)(y)}.$$

We have that the holonomy $\tilde{h} : \mathcal{W}(x) \rightarrow \mathcal{W}(y)$ along $\tilde{\mathcal{W}}_{\beta(\mathbf{t})}^s$ equals

$$\lim_{i \rightarrow \infty} (\alpha(\mathbf{b}_i)|\mathcal{W}(y))^{-1} \circ P_i \circ (\alpha(\mathbf{a}_i)|\mathcal{W}(x))$$

where P_i is a sequence of smooth maps from $\mathcal{W}(\alpha(\mathbf{a}_i)(x))$ to $\mathcal{W}(\alpha(\mathbf{b}_i)(y))$ converging to the identity. Using (4.1) and property (2) above we obtain

$$\lim_{i \rightarrow \infty} \alpha(\mathbf{a}_i)|\mathcal{W}(x) = \lim_{i \rightarrow \infty} H_{\alpha(\mathbf{a}_i)(x)} \circ H_x^{-1} = H_z \circ H_x^{-1} =: g_{xz}$$

since $\alpha(\mathbf{a}_i)(x) \rightarrow z$. Similarly

$$\lim_{i \rightarrow \infty} \alpha(\mathbf{b}_i)|\mathcal{W}(y) = \lim_{i \rightarrow \infty} H_{\alpha(\mathbf{b}_i)(y)} \circ H_y^{-1} = H_z \circ H_y^{-1} =: g_{yz}$$

since $\alpha(\mathbf{b}_i)(y) \rightarrow z$ also. So we get that

$$\tilde{h} = g_{yz}^{-1} \circ g_{xz} = H_y \circ H_x^{-1}.$$

Again using (4.1) and the definition of $\mu^{\mathcal{W},1}$ we get that $\alpha(\mathbf{a}_i)_* \mu_x^{\mathcal{W},1} = \mu_{\alpha(\mathbf{a}_i)(x)}^{\mathcal{W},1}$ and $\alpha(\mathbf{b}_i)_* \mu_y^{\mathcal{W},1} = \mu_{\alpha(\mathbf{b}_i)(y)}^{\mathcal{W},1}$.

So, finally, putting all together we get and sending i to infinity we get that $(g_{xz})_* \mu_x^{\mathcal{W},1} = \mu_z^{\mathcal{W},1}$ and $(g_{yz})_* \mu_y^{\mathcal{W},1} = \mu_z^{\mathcal{W},1}$ which gives that $\tilde{h}_* \mu_x^{\mathcal{W},1} = \mu_y^{\mathcal{W},1}$.

Now, since $\tilde{\mathcal{W}}_{\beta(\mathbf{t})}^s(x)$ is a graph over $\mathcal{W}_{\alpha(\mathbf{t})}^s(x)$ we get that ν_x^s -a.e. point correspond to μ_x^s -a.e. point and hence we get that for μ -a.e. point x and for μ_x^s -a.e. point $y \in \mathcal{W}_{\alpha(\mathbf{t})}^s(x)$ we get that $\tilde{y} = \alpha(\phi_x(y)\mathbf{w})(y)$ is a typical point for ν_x^s in $\tilde{\mathcal{W}}_{\beta(\mathbf{t})}^s(x)$ and hence the holonomy $\tilde{h} : \mathcal{W}(x) \rightarrow \mathcal{W}(\tilde{y})$ makes $\tilde{h}_* \mu_x^{\mathcal{W},1} = \mu_{\tilde{y}}^{\mathcal{W},1}$. So the proof finishes since we have that $h : \mathcal{W}(x) \rightarrow \mathcal{W}(y)$ equals $h = (\alpha(\phi_x(y)\mathbf{w}))^{-1} \circ \tilde{h}$ and we have that $\mu^{\mathcal{W}}$ is α invariant modulo multiplication by a constant. \square

4.3. Proof of Theorem 2.9. Take \mathbf{t} and \mathbf{s} as in Theorem 2.8. Take now a neighborhood of \mathbf{t} such that positive Lyapunov exponents other than χ in this neighborhood are all bigger than χ . For \mathbf{s} in this neighborhood, we call the strong unstable foliation associated to the positive Lyapunov exponents different from χ , V^{u-1} that does not depend on \mathbf{s} . Pick \mathbf{s} in this neighborhood and observe that if \mathbf{s} is on one side of $\ker \chi$, where $\chi(\mathbf{s}) > 0$,

(denote this side L^+) there is only one more positive Lyapunov exponent, i.e. χ , and no new exponent appears to the other side L^- , where $\chi(\mathbf{s}) < 0$. So, for $\mathbf{s} \in L^- \cup \ker \chi$, $V^{u-1} = \mathcal{W}_{\alpha(\mathbf{s})}^u$ and for $\mathbf{s} \in L^+$, $V^{u-1} \subsetneq \mathcal{W}_{\alpha(\mathbf{s})}^u$.

By Corollary 3.9 we see that the map $\mathbf{s} \rightarrow h(\xi_{u-1}, \alpha(\mathbf{s}))$ is linear in this neighborhood.

On the other hand, by Ledrappier-Young entropy formula we have that for $\mathbf{s} \in L^+$,

$$h_\mu(\alpha(\mathbf{s})) = h_\mu(\xi_u, \alpha(\mathbf{s})) = h(\xi_{u-1}, \alpha(\mathbf{s})) + \gamma_u \chi(\mathbf{s})$$

where ξ_u is any partition subordinated to $\mathcal{W}_{\alpha(\mathbf{s})}^u$ and γ_u does not depend on \mathbf{s} by its definition and the assumption on \mathbf{s} , see the definition of γ_u in subsection 3.4.

Finally, for $\mathbf{s} \in L^- \cup \ker \chi$, \mathbf{s} close to \mathbf{t} we have that

$$h_\mu(\alpha(\mathbf{s})) = h(\xi_{u-1}, \alpha(\mathbf{s})).$$

Hence we have that on L^+ , $h_\mu(\alpha(\mathbf{s}))$ is linear and on L^- , $h_\mu(\alpha(\mathbf{s}))$ is also linear. So, in order that this h_μ be differentiable its is necessary and sufficient that this two linear maps coincide. But since $\mathbf{s} \rightarrow h(\xi_{u-1}, \alpha(\mathbf{s}))$ is linear in the whole neighborhood, this is the same as asking that $\gamma_u = 0$. Hence, h_μ is differentiable at \mathbf{t} if and only if $\gamma_u = 0$.

Now, if conditional measures along \mathcal{W} are atomic, then we have by Theorem 2.8 that conditional measure along V^u is supported in V^{u-1} and hence $\delta^u = \delta^{u-1}$ which gives $\gamma^u = 0$ and hence h_μ is differentiable at \mathbf{t} . \square

5. PROOF OF THEOREMS 2.1 AND 2.3

5.1. Proof of Theorem 2.1. By Theorem 2.7 we know that conditional measures along Lyapunov directions are either Lebesgue or atomic. We will show that if all conditionals are atomic, then conditional measures along stable manifolds of $\alpha(\mathbf{t})$ are atomic and hence $h_\mu(\mathbf{t}) = 0$.

Observe that there without loss of generality we may assume \mathbf{t} to be a regular element since for every \mathbf{t} one can find a regular element \mathbf{t}' whose stable foliation contains that of \mathbf{t} . Hence if $h_\mu(\mathbf{t}) > 0$ then $h_\mu(\mathbf{t}') > 0$.

We shall build a sequence of nested sub-foliations

$$\tilde{\mathcal{V}}_0 \supset \tilde{\mathcal{V}}_1 \supset \cdots \supset \tilde{\mathcal{V}}_n$$

where $\mathcal{W}_{\alpha(\mathbf{t})}^s = \tilde{\mathcal{V}}_0$, $\tilde{\mathcal{V}}_n(x) = \{x\}$ and each $\tilde{\mathcal{V}}_i$ is either equal to $\tilde{\mathcal{V}}_{i-1}$ or has one less dimension for a.e. x . We shall prove also that conditional measures on the leaves $\tilde{\mathcal{V}}_i$ are supported by single leaves of $\tilde{\mathcal{V}}_{i+1}$ and the theorem will follow.

Since \mathbf{t} is regular it belongs to a Weyl chamber that we denote by C_0 . Let γ be a curve which begins at \mathbf{t} , passes through every Lyapunov hyperplane and crosses each Lyapunov hyperplane only once at a point that does not lie on any other Lyapunov hyperplane. An examples of such a curve is the half-circle in a two-dimensional plane through \mathbf{t} that is in general position, i.e. intersects all Lyapunov hyperplanes along different lines.

Let us number C_0, C_1, \dots, C_n the Weyl chambers and χ_1, \dots, χ_n the Lyapunov exponents, in the order they appear.

The stable foliation does not change within a Weyl chamber so we shall denote $\mathcal{W}_{C_i}^s$ the stable foliation associated to this Weyl chamber, also the sign of a Lyapunov exponent does not change so we may denote this sign by $\chi_j(C_i)$. So, we define $\tilde{\mathcal{V}}_0 = \mathcal{W}_{C_0}^s$ and $\tilde{\mathcal{V}}_i = \tilde{\mathcal{V}}_{i-1} \cap \mathcal{W}_{C_i}^s$. Clearly $\tilde{\mathcal{V}}_i \subset \tilde{\mathcal{V}}_{i-1}$ and $\tilde{\mathcal{V}}_i$ is a nice foliation since it is an intersection of stable foliations.

When passing from C_{i-1} to C_i χ_i is the only Lyapunov exponent that changes sign. So, if $\chi_i(C_i) < 0$ then $\chi_i(C_{i-1}) > 0$ so that $\mathcal{W}_{C_i}^s \supset \mathcal{W}_{C_{i-1}}^s$ and hence $\tilde{\mathcal{V}}_i = \tilde{\mathcal{V}}_{i-1}$. On the other hand, if $\chi_i(C_i) > 0$ then $\chi_i(C_{i-1}) < 0$ and hence $\mathcal{W}_{C_i}^s \subsetneq \mathcal{W}_{C_{i-1}}^s$ and in this case $\tilde{\mathcal{V}}_i \subsetneq \tilde{\mathcal{V}}_{i-1}$, in fact \mathcal{W}^i is no more in $\tilde{\mathcal{V}}_i$, i.e. $\mathcal{W}^i(x) \cap \tilde{\mathcal{V}}_i(x) = \{x\}$. Moreover, if we take an element inside C_{i-1} but close to $\ker \chi_i$ we have that \mathcal{W}^i is the slowest direction in $\mathcal{W}_{C_{i-1}}^s$ while $\tilde{\mathcal{V}}_i$ is inside the fast direction in $\mathcal{W}_{C_{i-1}}^s$ (which is exactly $\mathcal{W}_{C_i}^s$).

Let us fix measurable partitions η_i subordinated to $\mathcal{W}_{C_i}^s$ such that elements are open subsets of the leaves mod 0, i.e. the conditional measures of the boundaries are equal to zero.

We shall choose those partitions in such a way that if $\mathcal{W}_{C_i}^s \supset \mathcal{W}_{C_{i-1}}^s$ then $\eta_i < \eta_{i-1}$ and if $\mathcal{W}_{C_i}^s \subset \mathcal{W}_{C_{i-1}}^s$ then $\eta_i > \eta_{i-1}$. Let $\xi_0 = \eta_0$ and define inductively the measurable partitions $\xi_i = \xi_{i-1} \vee \eta_i$. It is sufficient to prove that $\mu_{\xi_0}^x(\xi_n(x)) > 0$ for a.e. x since by construction $\xi_n = \varepsilon$. To this end we shall argue inductively and prove that $\mu_{\xi_{i-1}}^x(\xi_i(x)) > 0$ for a.e. x . Once we know this, we have that for any measurable set A

$$\mu_{\xi_i}^x(A) = \frac{\mu_{\xi_{i-1}}^x(A \cap \xi_i(x))}{\mu_{\xi_{i-1}}^x(\xi_i(x))},$$

and hence

$$\mu_{\xi_{i-1}}^x(\xi_n(x)) = \mu_{\xi_i}^x(\xi_n(x)) \mu_{\xi_{i-1}}^x(\xi_i(x)),$$

which gives

$$\mu_{\xi_0}^x(\xi_n(x)) = \prod_{i=1}^n \mu_{\xi_{i-1}}^x(\xi_i(x)) > 0.$$

When $\mathcal{W}_{C_i}^s \supset \mathcal{W}_{C_{i-1}}^s$ we have that $\xi_i = \xi_{i-1}$ and hence $\mu_{\xi_{i-1}}^x(\xi_i(x)) > 0$ trivially. So, let us assume that $\mathcal{W}_{C_i}^s \subsetneq \mathcal{W}_{C_{i-1}}^s$ and hence $\eta_i > \eta_{i-1}$. Since conditionals along \mathcal{W}^i are atomic, by Theorem 2.8, we conclude that $\mu_{\eta_{i-1}}^x(\eta_i(x)) > 0$ for a.e. x .

Finally, we prove that $\mu_{\xi_{i-1}}^x(\xi_i(x)) > 0$ for a.e. x using the following lemma with $X = \eta_{i-1}(x)$, $\eta = \eta_i$, $\xi_1 = \xi_{i-1}$ and $\xi_2 = \xi_i$.

Lemma 5.1. *Let ξ_1, ξ_2 and η be three measurable partitions of a Lebesgue space X . Let us assume that $\xi_2 = \xi_1 \vee \eta$. Let μ be a measure in X . If there is x such that $\mu(\eta(x)) > 0$, then $\mu_{\xi_1}^y(\xi_2(y)) = \mu_{\xi_1}^y(\eta(x))$ and $\mu_{\xi_1}^y(\eta(x)) > 0$ for μ a.e. y in $\eta(x)$ and hence $\mu_{\xi_1}^y(\xi_2(y)) > 0$ for a.e. $y \in \eta(x)$.*

Proof. The first equality is trivial since $\xi_2(y) = \xi_1(y \cap \eta(y))$ and $\eta(x) = \eta(y)$ for $y \in \eta(x)$. For the inequality, let $D = \{y \in X : \mu_{\xi_1}^y(\eta(x)) = 0\}$. Observe that D is ξ_1 -saturated. Let $B = D \cap \eta(x)$. We want to show that $\mu(B) = 0$. We have that

$$\mu(B) = \int \mu_{\xi_1}^y(B) d\mu = \int_D \mu_{\xi_1}^y(B) d\mu + \int_{D^c} \mu_{\xi_1}^y(B) d\mu.$$

The first integral in the right-hand side is 0 since $B \subset \eta(x)$ and $\mu_{\xi_1}^y(\eta(x)) = 0$ for $y \in D$. The second is zero since for $y \in D^c$ we have that $\xi_1(y) \subset D^c$ and hence, since $B \subset D$ we get that $\xi_1(y) \cap D = \emptyset$ \square

5.2. Proof of Theorem 2.3. Theorem 2.3 is an immediate corollary of Theorem 2.9 and the following general criterion of absolute continuity.

Theorem 5.2. *Let $f : M \rightarrow M$ be a $C^{1+\theta}$ diffeomorphism preserving an ergodic measure μ . Let $TM = E^u \oplus E^c \oplus E^s$ be the Oseledec splitting associated to μ . Let us assume that:*

- (1) *E^c is tangent to a smooth foliation \mathcal{O} , that $Df|_{E^c}$ is an isometry w.r.t. to the standard metric in M and that conditional measures along \mathcal{O} are Lebesgue measure,*
- (2) *$E^u = E_1 \oplus \dots \oplus E_u$, $E^s = E_s \oplus \dots \oplus E_r$, where $\chi_i < \chi_j$ if $i < j$,*
- (3) *each E_i is tangent to an absolutely continuous Lyapunov foliation \mathcal{W}^i and conditional measures along \mathcal{W}^i are absolutely continuous w.r.t. Lebesgue for a.e. point.*

Then μ is absolutely continuous w.r.t. Lebesgue.

Proof. The proof reduces to see that conditional measure along stables and unstables are absolutely continuous. We shall argue by induction on the flag tangent to $E_1, E_1 \oplus E_2, \dots, E_1 \oplus \dots \oplus E_u = E^u$. So, let us call V_i the "foliation" tangent to $E_1 \oplus \dots \oplus E_i$. That conditional measures along $V_1 = \mathcal{W}^1$ are absolutely continuous is by assumption. Let us see that conditionals along V_2 is absolutely continuous and then the general step will follow as well.

Let R be the set of regular points. Take a regular point x and a zero Lebesgue measure set $A \subset V_2(x)$. We want to see that $\mu_{V_2}^x(A) = 0$ also. Taking $A \cap R$ we have that $\mu_{V_2}^x(A) = \mu_{V_2}^x(A \cap R)$ and $Leb_{V_2(x)}(A \cap R) = 0$. So, we may assume without loss of generality that A consists of regular points (indeed arguing similarly we may also assume A is inside some Pesin set if necessary). Now, since the foliation $\mathcal{W}^1 = V_1$ is absolutely continuous and indeed it is also absolutely continuous when restricted to $V_2(x)$, we may saturate the set A by V_1 leaves and get a set of 0 $Leb_{V_2(x)}$ -measure which is V_1 -saturated and which contains A . Let us call this set by B and let us see that $\mu_{V_2}^x(B) = 0$.

Now, if $\mu_{V_2}^x(B) > 0$ there should be a regular point $z \in V_2(x)$ such that $\mu_{V_2}^z(B) > 0$. But since $\mu_{V_2}^z$ is equivalent to Lebesgue measure this will imply that $Leb_{V_2(z)}(B) > 0$ and again, absolute continuity of V_1 and the

fact that B is V_1 saturated will imply that $Leb_{V_2(x)}(B) > 0$ which is a contradiction . \square

6. PROOF OF THEOREM 2.4

6.1. General facts about entropy. We shall make use of the following standard facts. Given measurable partitions ξ , η and ζ we have

- (1) $H(\xi \vee \eta | \zeta) = H(\xi | \zeta) + H(\eta | \zeta \vee \xi)$
- (2) If $\xi > \zeta$ then $H(\xi | \eta) \geq H(\zeta | \eta)$ and $H(\eta | \xi) \leq H(\eta | \zeta)$
- (3) If $\xi_n \uparrow \xi$ then $H(\xi_n | \eta) \uparrow H(\xi | \eta)$
- (4) If $\xi_n \downarrow \xi$ and $H(\xi_1 | \eta) < \infty$ then $H(\xi_n | \eta) \downarrow H(\xi | \eta)$
- (5) If $\eta_n \uparrow \eta$ and $H(\xi | \eta_1) < \infty$ then $H(\xi | \eta_n) \downarrow H(\xi | \eta)$
- (6) If $\eta_n \downarrow \eta$ then $H(\xi | \eta_n) \uparrow H(\xi | \eta)$
- (7) For every $n \in \mathbb{Z}$, $h(\xi \vee \eta, T) = h(\xi \vee T^n \eta, T)$

Given a measurable partition η we denote $\eta_T = \bigvee_{n \in \mathbb{Z}} T^n \eta$ and $\eta^+ = \bigvee_{n=0}^{\infty} T^n \eta$.

Lemma 6.1. *Given two measurable partitions ξ and η ,*

$$h(\xi \vee \eta, T) \leq h(\eta, T) + h(\xi \vee \eta_T, T).$$

Proof. First of all we have that

$$\begin{aligned} h(\xi \vee \eta, T) &= H(T^{-1} \xi \vee T^{-1} \eta | \xi^+ \vee \eta^+) \\ &= H(T^{-1} \eta | \xi^+ \vee \eta^+) + H(T^{-1} \xi | \xi^+ \vee T^{-1} \eta^+) \end{aligned}$$

Then we have that

$$\begin{aligned} h(\xi \vee T^{-n} \eta, T) &= H(T^{-1-n} \eta | \xi^+ \vee T^{-n} \eta^+) + H(T^{-1} \xi | \xi^+ \vee T^{-1-n} \eta^+) \\ &= H(T^{-1} \eta | T^n \xi^+ \vee \eta^+) + H(T^{-1} \xi | \xi^+ \vee T^{-1-n} \eta^+) \\ &\leq H(T^{-1} \eta | \eta^+) + H(T^{-1} \xi | \xi^+ \vee T^{-1-n} \eta^+) \end{aligned}$$

On one hand, the last term is bounded by $H(T^{-1} \xi | \xi^+) = h(\xi, T)$ and also $\xi \vee T^{-1-n} \eta^+ \uparrow \xi \vee \eta_T$ hence this last term $\downarrow H(T^{-1} \xi | \xi^+ \vee \eta_T)$. On the other hand, since η_T is a T -invariant partition we get easily that $h(\xi \vee \eta_T, T) = H(T^{-1} \xi | \xi^+ \vee \eta_T)$ and hence the inequality. \square

Finally we have the standard formula: given an invariant partition ζ (i.e. $p^{-1} \zeta = \zeta$) we have that

$$h(T) = h(T | \zeta) + \sup_{\xi} h(\xi \vee \zeta, T)$$

where $h(T | \zeta) = \sup_{\eta < \zeta} h(\eta, T)$.

Proposition 6.2. *Let us consider $T : (X, \mu) \rightarrow (X, \mu)$, $S : (Y, \lambda) \rightarrow (Y, \lambda)$ and assume that S is a factor of T , via a measure preserving map $p : (X, \mu) \rightarrow (Y, \lambda)$. Let ξ be a full entropy partition for T and η a partition such that $\eta_S = \varepsilon = \text{partition into points}$ and $p^{-1} \eta < \xi$. Then η is a full entropy partition for S .*

Proof. Let us call ζ the partition into pre-images of p . We have on one hand

$$h(T) = h(T|\zeta) + \sup_{\gamma} h(\gamma \vee \zeta, T) = h(S) + \sup_{\gamma} h(\gamma \vee \zeta, T)$$

and on the other hand since $(p^{-1}\eta)_T = p^{-1}(\eta_S) = p^{-1}\varepsilon = \zeta$:

$$\begin{aligned} h(T) &= h(\xi, T) = h(\xi \vee p^{-1}\eta, T) \leq h(p^{-1}\eta, T) + h(\xi \vee (p^{-1}\eta)_T, T) \\ &= h(\eta, S) + h(\xi \vee \zeta, T) \end{aligned}$$

where the inequality follows from Lemma 6.1. Thus $h(\eta, S) = h(S)$ i.e. η is a full entropy partition. Observe also that ξ is also a full entropy partition for the fiber-entropy $\sup_{\gamma} h(\gamma \vee \zeta, T)$. \square

6.2. Matching of Lyapunov half-spaces. Here we assume α and α_0 are \mathbb{Z}^k actions on an infranilmanifold M as in Theorem 2.4.

Unstable foliation $W_{\alpha_0(\mathbf{m})}^u$ for an element of the algebraic action α_0 is right homogeneous. Lyapunov foliations W^i that are one-dimensional under our assumptions are intersections of unstable foliations of different elements and are projections of cosets of one-parameter subgroups in N .

Let $p : M \rightarrow M$ be the semiconjugacy between these actions and let μ be an ergodic large measure invariant by α . We want to prove that Weyl chambers for both actions match. We do this in two steps. First we prove a general result which requires no assumption on the linear action:

Lemma 6.3. *If L is a Lyapunov hyperplane for α_0 then L is also a Lyapunov hyperplane for α and Lyapunov half-spaces match.*

Remark 6. Observe that this lemma implies that the number of non-proportional (and hence nonzero) Lyapunov exponents for the nonlinear action is greater or equal than the number of coarse Lyapunov distributions for the linear action. Then later we prove that under our assumptions Lyapunov hyperplanes for the nonlinear action also correspond to Lyapunov hyperplanes for the linear action.

Proof of Lemma 6.3. Assume by contradiction that there are two elements $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^k$ on different sides of L but in the same Weyl chamber for α . Then we will have that $\mathcal{W}_{\alpha(\mathbf{n})}^u(x) = \mathcal{W}_{\alpha(\mathbf{m})}^u(x)$ but $W_{\alpha_0(\mathbf{n})}^u \neq W_{\alpha_0(\mathbf{m})}^u$. Since the semiconjugacy maps unstable manifolds for α into unstable manifolds for α_0 , either for \mathbf{n} or \mathbf{m} the following is true (we assume it is for \mathbf{n}):

$$p(\mathcal{W}_{\alpha(\mathbf{n})}^u(x)) \subset p(x)W_{\alpha_0(\mathbf{n})}^u \cap E_{\alpha_0(\mathbf{m})}^u \subsetneq p(x)W_{\alpha_0(\mathbf{n})}^u.$$

Now, we may take a full entropy increasing partition $\tilde{\xi}$ for $\alpha(\mathbf{n})$ subordinated to $\mathcal{W}_{\alpha(\mathbf{n})}^u$ like the one built in [14] and an increasing partition η subordinated to $yW_{\alpha_0(\mathbf{n})}^u \cap W_{\alpha_0(\mathbf{m})}^u$, again like in [14], and we may build them in such a way that $p^{-1}\eta < \tilde{\xi}$.

Since the negative iterates of $\alpha_0(\mathbf{n})$ contract $W_{\alpha_0(\mathbf{n})}^u \cap W_{\alpha_0(\mathbf{m})}^u$ we have that $\eta_{\alpha_0(\mathbf{n})} = \varepsilon$ and hence $(p^{-1}\eta)_{\alpha(\mathbf{n})} = \zeta$. So we have that, using Proposition 6.2, η is a full entropy partition for $\alpha_0(\mathbf{n})$.

But since we are considering Lebesgue measure for α_0 , $W_{\alpha_0(\mathbf{n})}^u \cap W_{\alpha_0(\mathbf{m})}^u \subsetneq E_{\alpha_0(\mathbf{n})}^u$ and η is subordinated to $yW_{\alpha_0(\mathbf{n})}^u \cap W_{\alpha_0(\mathbf{m})}^u$ we have that

$$h(\eta, \alpha_0(\mathbf{n})) < h(\alpha_0(\mathbf{n}))$$

which gives a contradiction. \square

Now, we consider the case at hand: there are no proportional Lyapunov exponents for the linear action, i.e. there are different exponents χ_1, \dots, χ_n and none of them are proportional. Thus in this case α also has n different non-proportional Lyapunov exponents and that the Weyl chambers coincides. In particular we have that there are positive numbers c_i such that $\tilde{\chi}_i = c_i \chi_i$ for $i = 1, \dots, n$. So, we have already excluded zero exponents.

Corollary 6.4. *For every $i = 1, \dots, n$ there is a Lyapunov foliation \mathcal{W}^i associated to $\tilde{\chi}_i$, such that the leaf $\mathcal{W}^i(x)$ is mapped by the semiconjugacy p into the corresponding coset $W^i(p(x))$.*

6.3. Conclusion of the proof.

Proposition 6.5. *Conditional measures along \mathcal{W}^i are equivalent to Lebesgue a.e.*

Proof. We shall argue as in the proof of preservation of Weyl chambers. Using Theorem 2.7 and arguing by contradiction we may assume that conditional measures along \mathcal{W}^i are atomic a.e. Then take \mathbf{t} and \mathbf{s} as in Theorem 2.8 for the suspended action and we may take $\mathbf{s} \in \mathbb{Z}^k$. We will use the same notation α and α_0 for the suspended actions of \mathbb{R}^k . Take now an $\alpha(-\mathbf{s})$ increasing partition ξ subordinated to $\mathcal{W}_{\alpha(\mathbf{t})}^s(x) \subsetneq \mathcal{W}_{\alpha(\mathbf{s})}^s(x)$. Then, since $p(\mathcal{W}_{\alpha(\mathbf{t})}^s(x)) \subset p(x)W_{\alpha_0(\mathbf{t})}^s \subsetneq p(x)W_{\alpha_0(\mathbf{s})}^s$ we can build another partition η subordinated to $p(x)W_{\alpha_0(\mathbf{t})}^s$ such that $p^{-1}\eta < \xi$. Since by Theorem 2.8 conditional measure along $\mathcal{W}_{\alpha(\mathbf{s})}^s(x)$ is in fact supported in $\mathcal{W}_{\alpha(\mathbf{t})}^s(x)$ we have that ξ is a full entropy partition for $\alpha(-\mathbf{s})$ and then by Proposition 6.2 η should be also a full entropy partition for $\alpha_0(-\mathbf{s})$, but this is impossible since $W_{\alpha_0(\mathbf{t})}^s \subsetneq W_{\alpha_0(\mathbf{s})}^s$. \square

Now we can use Theorem 5.2 and conclude that μ is an absolutely continuous measure. This concludes the proof of Theorem 2.4(1).

Theorem 2.4(2) follows exactly as in [7]. Or, more precisely, it is proven there using information that we already possess. Namely [7, Lemma 4.4] asserts that the semiconjugacy restricted to a.e. leaf of a Lyapunov foliation is a diffeomorphism. Hence it matches asymptotic rates of expansion/contraction along the foliations and thus Lyapunov exponents. \square

7. PROOF OF THEOREM 2.5

7.1. Uniqueness. For the proof of uniqueness in Theorem 2.5 we will use the invariant affine structures on stable manifolds of the action α . We shall prove that affine structures for unstable manifolds of the nonlinear action

α are intertwined by the semiconjugacy with the standard affine structure of unstable spaces for the linear action α_0 . Notice that due to Theorem 2.4 no resonance condition holds for α . Existence of these affine structures is guaranteed by the non-resonance condition, see [6, Section 6.2].

Proposition 7.1. *For every $\mathbf{t} \in \mathbb{Z}^k$ and on each leaf of $\mathcal{W}_{\alpha(\mathbf{t})}^s$ there is a unique smooth α -invariant affine structure together with a frame such that for any regular point x and j such that $\chi_j(\mathbf{t}) < 0$ the one-dimensional leaf $\mathcal{W}^j(x)$ is a coordinate line in $\mathcal{W}_{\alpha(\mathbf{t})}^s(x)$ and for any regular point $z \in \mathcal{W}_{\alpha(\mathbf{t})}^s(x)$ the affine structure on $\mathcal{W}^j(z)$ coincides with the restriction of the affine structure on $\mathcal{W}_{\alpha(\mathbf{t})}^s(x)$.*

We will use additive notations for various invariant foliations associated with the action α_0 .

Proposition 7.2. *For almost every regular point z the restriction of the semiconjugacy h to the leaf $\mathcal{W}_{\alpha(\mathbf{t})}^s(z)$ is an affine bijection between $\mathcal{W}_{\alpha(\mathbf{t})}^s(z)$ and the hyperplane $p(z) + E_{\alpha_0(\mathbf{t})}^s$.*

Proof. Take z for which almost every point of the leaf $\mathcal{W}_{\alpha(\mathbf{t})}^s(z)$ with respect to the s -dimensional volume is regular. Since conditional measures are equivalent to Lebesgue, the set of such points is of full μ measure. Thus there is a dense subset of $\mathcal{W}^i(z)$ where leaves of \mathcal{W}^j for all $j \neq i$ are defined. By Proposition 7.1 any such manifold is a part of a corresponding line and its affine parameterization agrees with the one coming from the affine structure in $\mathcal{W}^i(z)$. But we already know that the semiconjugacy on any leaf of \mathcal{W}^j is affine. Furthermore for each regular $y \in \mathcal{W}^i(z)$ the manifold $\mathcal{W}^j(y)$ cannot be just an interval but must be the whole line in the affine structure. Thus we know that h restricted to $\mathcal{W}^i(z)$ is affine on a dense set of lines parallel to each coordinate direction. Hence it is affine. \square

Lemma 7.3. *For any point z satisfying the assertion of Proposition 7.2, the manifold $\mathcal{W}^i(z)$ is a complete manifold properly embedded into \mathbb{R}^n and at a bounded distance from E^i . Indeed, if we denote the semiconjugacy restricted to $\mathcal{W}^i(z)$ by $h_i^z : \mathcal{W}^i(z) \rightarrow h(z) + E^i$, then its inverse, $p_i^z : h(z) + E^i \rightarrow \mathcal{W}^i(z)$ is a proper diffeomorphism at a bounded distance from the inclusion.*

Proof. Proposition 7.2 implies this statement for any compact part of $\mathcal{W}^{i,+}(z)$. But since h is a bounded distance away from identity for any sequence of points on the $h(z) + E^i$ which goes to infinity the pre-images go to infinity too. The assertion of about the inverse of the semiconjugacy follows from the fact that the semiconjugacy is at a bounded distance from identity. \square

Now we shall show how the Hopf argument applies in this case to get uniqueness similar to what is done for instance in [16]. To this end we will need that for any two given regular points x_1, x_2 (possibly regular with respect to different large measures), the stable manifold of one intersects the unstable manifold of the other. This is done through an index argument.

Lemma 7.4. *Let $E^i \subset \mathbb{R}^n$, $i = 1, 2$ be two subspaces such that $E^1 \oplus E^2 = \mathbb{R}^n$. Let $p_i : E^i \rightarrow \mathbb{R}^n$, $i = 1, 2$ be two proper embeddings at a bounded distance from inclusion. Call $p_i(E^i) = W_i$, $i = 1, 2$. Then $W_1 \cap W_2 \neq \emptyset$.*

Proof. Let us assume by contradiction that $W_1 \cap W_2 = \emptyset$. Let D_i be closed unit disks in E_i and define for $0 < t \leq 1$,

$$X_t : D_1 \times D_2 \rightarrow S^{n-1} \subset E^1 \oplus E^2$$

by

$$X_t(v_1, v_2) = \frac{p_1(v_1/t) - p_2(v_2/t)}{\|p_1(v_1/t) - p_2(v_2/t)\|}.$$

Observe that X_t is well defined since the denominator is never 0. Let us write $p_i(z) = z + \psi_i(z)$, we have that there is $C > 0$ such that $\|\psi_i(z)\| \leq C$ for every $z \in E^i$.

Let us see that as $t \rightarrow 0$ we have that X_t restricted to $\partial(D_1 \times D_2)$ converges uniformly to

$$X_0(v_1, v_2) = \frac{v_1 - v_2}{\|v_1 - v_2\|}.$$

Indeed

$$p_1(v_1/t) - p_2(v_2/t) = \frac{v_1 - v_2}{t} + \psi_1(v_1/t) - \psi_2(v_2/t),$$

hence

$$X_t(v_1, v_2) = \frac{p_1(v_1/t) - p_2(v_2/t)}{\|p_1(v_1/t) - p_2(v_2/t)\|} = \frac{v_1 - v_2 + t(\psi_1(v_1/t) - \psi_2(v_2/t))}{\|v_1 - v_2 + t(\psi_1(v_1/t) - \psi_2(v_2/t))\|}$$

since the ψ_i are uniformly bounded and the denominator is bounded away from zero when $(v_1, v_2) \in \partial(D_1 \times D_2)$ and t is small, we get $X_t \rightarrow X_0$ uniformly.

But then it is known that X_0 is a map of nonzero degree (it is a homeomorphism), while X_1 restricted to $\partial(D_1 \times D_2)$ should have zero degree since it extends to $D_1 \times D_2$.

□

Now, take μ_1 and μ_2 two ergodic large measures. Fix an element of the action $f := \alpha(\mathbf{n})$ with all exponents nonzero. We shall prove uniqueness using f . Let us call G the set of points satisfying the conclusion of Proposition 7.2, we have that G has full measure for every large measure.

Take a continuous function ϕ , we will prove that $\int \phi d\mu_1 = \int \phi d\mu_2$. Let us take a set $B_1 \subset G$ of full μ_1 measure such that for $x \in B_1$, $\phi^+(x) = \phi^-(x) = \int \phi d\mu_1$, here ϕ^+ and ϕ^- denote forward and backward Birkhoff averages (with respect to f) respectively. Similarly take a set $B_2 \subset G$ of full μ_2 measure where $\phi^+(x) = \phi^-(x) = \int \phi d\mu_2$ for $x \in B_2$. Now take sets $A_i \subset B_i$ of full μ_i measure such that if a point x is in A_i then $\text{Leb}_{W^u(x)}$ almost every point y in $W^u(x)$ is in B_i . We have that A_i have full measure by the absolute continuity of the stable and unstable foliations.

We know that ϕ^+ is constant on stable manifolds and ϕ^- is constant on unstable manifolds.

We now lift all the objects to the universal covering in order to define holonomy maps in a more clear manner, we denote points in the universal covering and in the manifold in the same manner and this should not give any confusion. Take now two points $x_1 \in A_1$ and $x_2 \in A_2$. By Lemmas 7.3 and 7.4 we have that $W^s(x_1) \cap W^u(x_2) \neq \emptyset$, but we do not know a priori how this intersection is. Now, the semiconjugacy must send this intersection into the intersection of $h(x_1) + E^s$ and $h(x_2) + E^u$ which is a point. By Proposition 7.2, we know that the semiconjugacy restricted to $W^u(x_2)$ is one to one, hence this intersection is a point. Still we do not know if this intersection is transversal, so we can not follow the usual Hopf argument. In any case, since almost every point in $W^u(x_1)$ is in G we can define the holonomy map $\pi : W^u(x_1) \rightarrow W^u(x_2)$ by $\pi(z) = W^s(z) \cap W^u(x_2)$. Observe that π is a priori only defined on a set of full Lebesgue measure in $W^u(x_1)$. We want to prove that π is absolutely continuous but since the intersection defining π is not transversal a priori we can not follow the usual absolute continuity proof. What we have is that the semiconjugacy restricted to $W^u(x_i)$ is smooth, in fact it is affine with respect to the affine structure and the semiconjugacy also conjugates the holonomies, that is: if we define $Hol : h(x_1) + E^u \rightarrow h(x_2) + E^u$ as we did with π we have that $h \circ \pi = Hol \circ h$ for every point in $W^u(x_1)$ where π is defined. But Hol is a smooth map since Hol is simply a translation also h restricted to $W^u(x_i)$ is smooth hence $\pi = h^{-1} \circ Hol \circ h$ coincides a.e. with a smooth map and hence it is absolutely continuous. Now we have that $B_1 \cap W^u(x_1)$ has full Lebesgue measure in $W^u(x_1)$ and hence its image by π has also full Lebesgue measure in $W^u(x_2)$ and hence this image intersects B_2 , that is, we can take a point $a \in B_1$ whose stable manifold contains a point $b \in B_2$ hence we have that $\int \phi d\mu_1 = \phi^+(x_1) = \phi^+(x_2) = \int \phi d\mu_2$ and we are done.

7.2. Semiconjugacy and measurable isomorphism of α and α_0 . Let us see that the semiconjugacy is one to one over a set of full measure. Let R be the set of regular points satisfying the conclusion of Lemma 7.3. We shall see that the restriction of h to R is one to one. Let us fix a nonsingular element of the action. We already know that the restriction of h to stable and unstable manifolds of regular points is a diffeomorphism. Take x and y in R and assume by contradiction that $h(x) = h(y) = a$. By Lemmas 7.3 and 7.4 we know that $W^s(x) \cap W^u(y) \neq \emptyset$. Take z in this intersection. Then $h(z) \in (h(x) + E^s) \cap (h(y) + E^u)$ but since $h(x) = h(y) = a$ this last intersection is a and hence $h(z) = a$. Now, injectivity along stable manifolds gives a contradiction since $z \in W^s(x)$ and $h(z) = a = h(x)$. Since the image of R has full measure, we get that the restriction of h to R gives a measurable isomorphism between α and α_0 and thus we finish the proof of Theorem 2.5.

8. PROOF OF THEOREM 2.6

We first need to describe properly the classes of cocycles considered in Theorem 2.6. Let us fix a small positive number ε and consider Pesin sets $\mathfrak{R}_\varepsilon^l$ as defined in (3.1).

Let us consider a Lyapunov Riemannian metric on each Lyapunov distribution defined on the set of full measure $\mathfrak{R}_\varepsilon = \bigcup_l \mathfrak{R}_\varepsilon^l$. It is defined similarly to (3.2) with summation over \mathbb{Z}^k instead of integration. By [9, Proposition 5.3] this metric is Hölder continuous on each $\mathfrak{R}_\varepsilon^l$. Now consider a system of neighborhoods $P_\varepsilon(x)$, sometimes called *Pesin boxes*, of points in \mathfrak{R}_ε whose size depends on l and slowly oscillates with the action, similarly to the function K_ε from Proposition 3.1. Using a local coordinate system from a fixed finite atlas project the Lyapunov metric from T_x to the Pesin box around x with constant coefficients. Thus we obtain a system of locally defined metrics.

Definition 2. A cocycle β defined on \mathfrak{R}_ε is called *Lyapunov Hölder* if for any l , $x \in \mathfrak{R}_\varepsilon^l$ β is Hölder continuous on $\mathfrak{R}_\varepsilon^l \cap P_\varepsilon(x)$ with Hölder exponent and constant independent of x and l .

Similarly we define Lyapunov smooth cocycles by requiring smoothness along local stable manifolds of points in $\mathfrak{R}_\varepsilon^l$ with uniform bounds on derivative with respect to a Lyapunov metric within Pesin boxes.

Notice that by Proposition 7.1 the semi-conjugacy h between α and the linear action α_0 is bijective on an increasing sequence of compact Pesin sets as well on stable and unstable manifolds of points from those sets with respect to all elements of the action α . The strategy of the proof is to use these bijections to construct cocycles over α_0 and then use the method of [10].

Take the image of a Pesin set \mathcal{P} under the semi-conjugacy. If a solution of the coboundary equation exists then along the stable manifold \mathcal{W} of any element of the action is given by the familiar telescoping sum see e.g. [10, Proof of Theorem 3.1]. This implies in particular that the solution (transfer function) is Lyapunov Hölder or Lyapunov smooth if the cocycle has one of those properties.

By the absolute continuity $\mathcal{W} \cap \mathcal{P}$ has large conditional measure in \mathcal{W} and the union of our Pesin sets has full conditional measure. Now one considers periodic cycles anchored at points of the Pesin sets. Any two successive points in such a cycle lie on a one-dimensional Lyapunov line and any three successive points lie in a stable manifold of some element. The last statement follows from the TNS condition that is weaker than our strongly simple property. One can simply consider the situation after the semi-conjugacy, as a cocycle over the linear action. Arguing as in [10] we deduce that solution can be constructed consistently from a single typical point to the union of Pesin sets which has full measure. Since the semi-conjugacy is bijective on a full measure set and is smooth along almost

every stable manifold the solution can be brought back, and, as we pointed out, is then Lyapunov Hölder or Lyapunov smooth. \square

Remark 7. In the absence of semi-conjugacy but assuming strongly simple and no resonance conditions one can still extend the solution along Lyapunov lines but due to the “holes” in the union of Pesin sets the argument works only locally. This leads to the following statement.

Let μ be a measure as in Theorem 2.3. The spaces of classes of Lyapunov Hölder (corr. Lyapunov smooth) cocycles with respect to cohomology with Lyapunov Hölder (corr. Lyapunov smooth) transfer functions are finite dimensional.

Even in the absence of such holes the solution can be constructed on the universal cover but cannot in general be projected to the original manifold since the possibility of the action preserving a non-trivial homology class cannot be excluded.

9. BEYOND THE STRONGLY SIMPLE CASE

9.1. Summary. We can tentatively claim partial generalizations of the some results of this paper in the presence of multiple or positively proportional (but not negatively proportional) exponents. We can also outline the limits of applicability for our methods and formulate plausible conjectures.

One should consider separately the general case of hyperbolic measures for smooth actions as in Section 2.1 and large measures for actions on tori and nilmanifolds with hyperbolic homotopy data as in Section 2.2. At the level of linear algebra three effects may appear separately or in combinations:

- (1) *Negatively proportional exponents.* Our methods that are essentially geometric are not suitable for this situation. The main problem with using the linear algebra of Lyapunov exponents is that in the representative symplectic case the picture of Lyapunov hyperplanes and Weyl chambers is the same as for the product of rank one actions where rigidity does not take place. Thus there is not much hope for developing a general theory along the lines of [9].

Even for algebraic actions on a torus measure rigidity is established by different methods that take into account global Diophantine properties of stable foliations [4]. Another approach can be developed based on an unpublished preprint of J. Feldman and M. Smorodinsky from the early 1990s. Finding a non-uniform version of these arguments is a serious albeit not a hopeless challenge.

Hyperbolic measures without negatively proportional Lyapunov exponents are called *totally non-symplectic* (TNS).

- (2) *Multiple exponents.* The first central step of our approach is “freezing” the action in question along the walls of Weyl chambers. Notice that for linear (and hence algebraic) actions this is possible in the semisimple case, i.e. in the absence of Jordan blocks. For actions on

tori and nilmanifolds assuming that the algebraic model (the homotopy data) is semisimple helps. In the presence of jordan blocks the situation is less hopeful.

- (3) *Simple positively proportional exponents.* This case (assuming that no other effects appear) is the most hopeful and is discussed in more detail below. The key issue here is understanding resonances and invariant geometric structures that appear on coarse Lyapunov foliations.

9.2. Hyperbolic measures for actions on manifolds. The main difficulty here is that vanishing of a Lyapunov exponent does not guarantee that along Lyapunov foliations (even if the exponent is simple and if those exist) on a set of large measure the distances remain bounded. The technical device that allows to overcome this problem in the strongly simple case is the synchronizing time change described in Section 3.3. This is easily modified to obtain bounded growth estimates like in Proposition 3.4 for any one Lyapunov direction, e.g. the fastest for which Lyapunov distribution is integrable. However, in general fin general time change would be different in general so no simultaneous “freezing” is possible. This problem looks fundamental and probably cannot be overcome within the usual $\text{rank} \geq 2$ assumption.

But in fact in our argument synchronization of one exponent is achieved along the whole Lyapunov hyperplane. If the rank is ≥ 3 that *simultaneous* synchronization of two exponents can be achieved along a codimension two subspace and so on. This of course is under the assumption that exponents are simple. Thus the following statement holds:

Simultaneous synchronization of all proportional exponents is possible if their number does not exceed the rank of the action minus one.

Now one considers an invariant geometric structure on the coarse Lyapunov foliation. In the absence of double resonances this structure is flat affine and one can show that Lyapunov distributions integrate to foliations into lines with respect to this structure. The critical π -partition argument holds in this case and allows to show that conditional measures on the coarse Lyapunov foliation are supported on an affine subspaces and invariant under transitive groups of affine transformations on those subspaces. Hence those conditional measures are either atomic or absolutely continuous on smooth submanifolds of the leaves of the coarse Lyapunov foliations.

The arguments outlined above lead to the proof of a *generalization of Theorem 2.1 for TNS actions with simple positively proportional exponents, no double resonances if the number of exponents proportional to a given one does not exceed the rank of the action minus one.*

Detailed proofs will appear in a subsequent paper.

The case with double resonances is somewhat more complicated because for the slow directions there are no unique curves tangent to the slow directions similar to like in the affine case. Instead there are some parametric families of such curves like parabolas in the case of $2 : 1$ resonance. If one can prove that Lyapunov distributions integrate to certain families of such curves the rest of the argument should be similar to the non-resonance case.

An extension of Theorem 2.3 looks more problematic. The problem is that the full entropy assumption does not catch contributions coming from different positively proportional exponent. One should look for an appropriate “high entropy” assumptions that would lead to the assertion that conditional measure along the coarse Lyapunov foliation is absolutely continuous. After that absolute continuity of the measure can be established, similarly to the proof of Theorem 2.3.

9.3. Actions on tori and nil-manifolds. As was mentioned above, our methods are restricted to the TNS case so we make this assumption for the algebraic action α_0 . To be able to carry out the “freezing” argument we also need to avoid Jordan blocks for the action α_0 , i.e. to assume that its linear part is semi-simple (diagonalizable over \mathbb{C}).

Then the action along the Lyapunov hyperplanes is an isometry. The main issue is to prove that there are no new Lyapunov hyperplanes for the action α . So far we can prove this in certain special cases, e.g. If new Lyapunov exponents for α not proportional to those of α_0 (and hence new Lyapunov hyperplanes) appear, corresponding Lyapunov foliations must collapse under the semi-conjugacy.

Entropy considerations like in Section 6.1 provide for that: collapsing of certain directions leads to entropy deficit although arguments become more involved.

After that one can follow the general line of arguments in Sections 6 and 7 to obtain *an extension of Theorem 2.5 to the TNS non-resonance case*. A particular case where double exponents are allowed due to existence of complex eigenvalues for α_0 is announced in [8]. Detailed proofs will appear in a subsequent paper.

Resonances both for α_0 and for α represent an additional difficulty but basically one should prove intertwining of geometric structures and hence smoothness of the semi-conjugacy along the coarse Lyapunov foliations. Thus one can formulate desired outcome as follows.

Conjecture. *Let α_0 be a totally non-symplectic \mathbb{Z}^k action by automorphisms of an infranilmanifold and α be an action with homotopy data α_0 . Then every large invariant measure for α is absolutely continuous and has the same Lyapunov characteristic exponents as α_0 .*

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